**Summation problem**

 Find

$$\sum\_{n=1}^{100}\frac{n}{n^{4}+4}$$



(1) Factorization

 $n^{4}+4=\left(n^{4}+4n^{2}+4\right)-4n^{2}=\left(n^{2}+2\right)^{2}-\left(2n\right)^{2}$

 $=\left(n^{2}+2n+2\right)\left(n^{2}-2n+2\right)$

(2) Partial Fraction

 $\frac{n}{n^{4}+4}=\frac{A}{n^{2}+2n+2}+\frac{B}{n^{2}-2n+2}$

 $n=A(n^{2}-2n+2)+B(n^{2}+2n+2)$

 $A=-\frac{1}{4}, B=\frac{1}{4}$

 Hence $\frac{n}{n^{4}+4}=\frac{1}{4}\left(\frac{1}{n^{2}-2n+2}-\frac{1}{n^{2}+2n+2}\right)=\frac{1}{4}\left[\frac{1}{\left(n-1\right)^{2}+1}-\frac{1}{\left(n+1\right)^{2}+1}\right]$

(3) Summation

 $\sum\_{n=1}^{100}\frac{n}{n^{4}+4}=\frac{1}{4}\sum\_{n=1}^{100}\left[\frac{1}{\left(n-1\right)^{2}+1}-\frac{1}{\left(n+1\right)^{2}+1}\right]$

 $=\frac{1}{4}\left\{\left[\frac{1}{0^{2}+1}-\frac{1}{2^{2}+1}\right]+\left[\frac{1}{1^{2}+1}-\frac{1}{3^{2}+1}\right]+\left[\frac{1}{2^{2}+1}-\frac{1}{4^{2}+1}\right]+…+\left[\frac{1}{97^{2}+1}-\frac{1}{99^{2}+1}\right]+\left[\frac{1}{98^{2}+1}-\frac{1}{100^{2}+1}\right]+\left[\frac{1}{99^{2}+1}-\frac{1}{101^{2}+1}\right]\right\}$

 $=\frac{1}{4}\left\{\frac{1}{0^{2}+1}+\frac{1}{1^{2}+1}-\frac{1}{100^{2}+1}-\frac{1}{101^{2}+1}\right\}$

 $=\frac{1}{4}\left\{1+\frac{1}{2}-\frac{1}{10001}-\frac{1}{10202}\right\}≈\overline{\overline{0.374950497501}}$

Further interest point:

$$\sum\_{n=1}^{\infty }\frac{n}{n^{4}+4}=\frac{3}{8}=0.375$$

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